Math 150, ***Lecture Notes- Bonds*** **Name____________**

Section 5.6 *Inverse Trigonometric Functions: Differentiation*

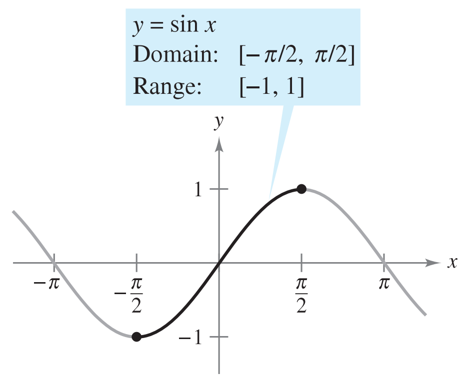
Inverse Trigonometric Functions

This section begins with a rather surprising statement: *None of the six basic trigonometric functions has an inverse function*. This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one. In this section you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the *restricted domains*.

In Example 4 of Section 5.3, you saw that the sine function is increasing (and therefore is one-to-one) on the interval $[-\pi/2, \pi/2]$ (see Figure 5.28). On this interval you can define the inverse of the *restricted* sine function as

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le \arcsin x \le \pi/2$.



The sine function is one-to-one on $[-\pi/2, \pi/2]$.
Figure 5.28

NOTE The term "iff" is used to represent the phrase "if and only if."

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the following definition.

Definitions of Inverse Trigonometric Functions

Function	*Domain*	*Range*
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{arccot}\, x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \text{arcsec}\, x$ iff $\sec y = x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi,\quad y \ne \frac{\pi}{2}$
$y = \text{arccsc}\, x$ iff $\csc y = x$	$\lvert x\rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\quad y \ne 0$

$f(x) = \sin(x)$

Angle θ → $-1 \le y \le 1$

NOTE The term "arcsin x" is read as "the arcsine of x" or sometimes "the angle whose sine is x." An alternative notation for the inverse sine function is "$\sin^{-1} x$."

$f^{-1}(x) = \sin^{-1}(x)$

The graphs of the six inverse trigonometric functions are shown in Figure 5.29.

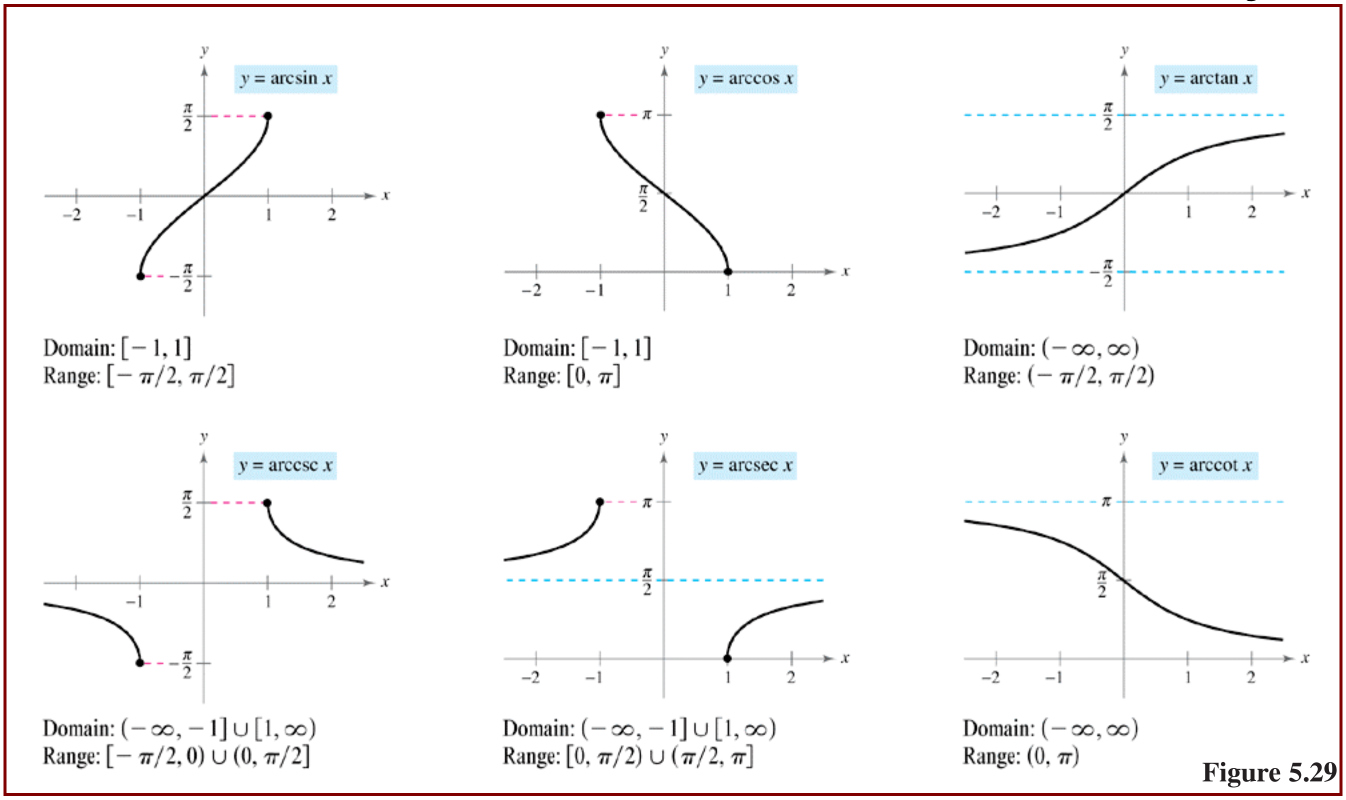


Figure 5.29

Ex.1 Evaluating Inverse Trigonometric Functions

Evaluate each function.

NOTE When evaluating inverse trigonometric functions, remember that they denote *angles in radian measure*.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

Let $\theta = \arcsin\left(-\frac{1}{2}\right)$

$\sin(\theta) = \sin\left[\arcsin\left(-\frac{1}{2}\right)\right]$

$\sin(\theta) = -\frac{1}{2}$

$\theta = -\frac{\pi}{6}$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Let

$\theta = \cos^{-1}(0)$

$\cos(\theta) = \cos[\cos^{-1}(0)]$

$\cos(\theta) = 0$

$\theta = \frac{\pi}{2}$

$\arccos(0) = \frac{\pi}{2}$

let

$\theta = \tan^{-1}(\sqrt{3})$

$\tan(\theta) = \sqrt{3}$

$\frac{\sin(\theta)}{\cos(\theta)} = \frac{\sqrt{3}}{1}$

$\frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$\theta = \frac{\pi}{3}$

$\frac{\pi}{3} = \arctan(\sqrt{3})$

Let

$\theta = \arcsin(0.3)$

$\sin(\theta) = 0.3$??

$\theta \approx 0.304693$ RAD

Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For x-values outside these domains, these two properties do not hold. For example, $\arcsin(\sin \pi)$ is equal to 0, not π.

Properties of Inverse Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

Ex.2 Solving an Equation

Solve $\quad \arctan(2x - 3) = \dfrac{\pi}{4}$

$$\tan[\arctan(2x-3)] = \tan\left(\frac{\pi}{4}\right)$$

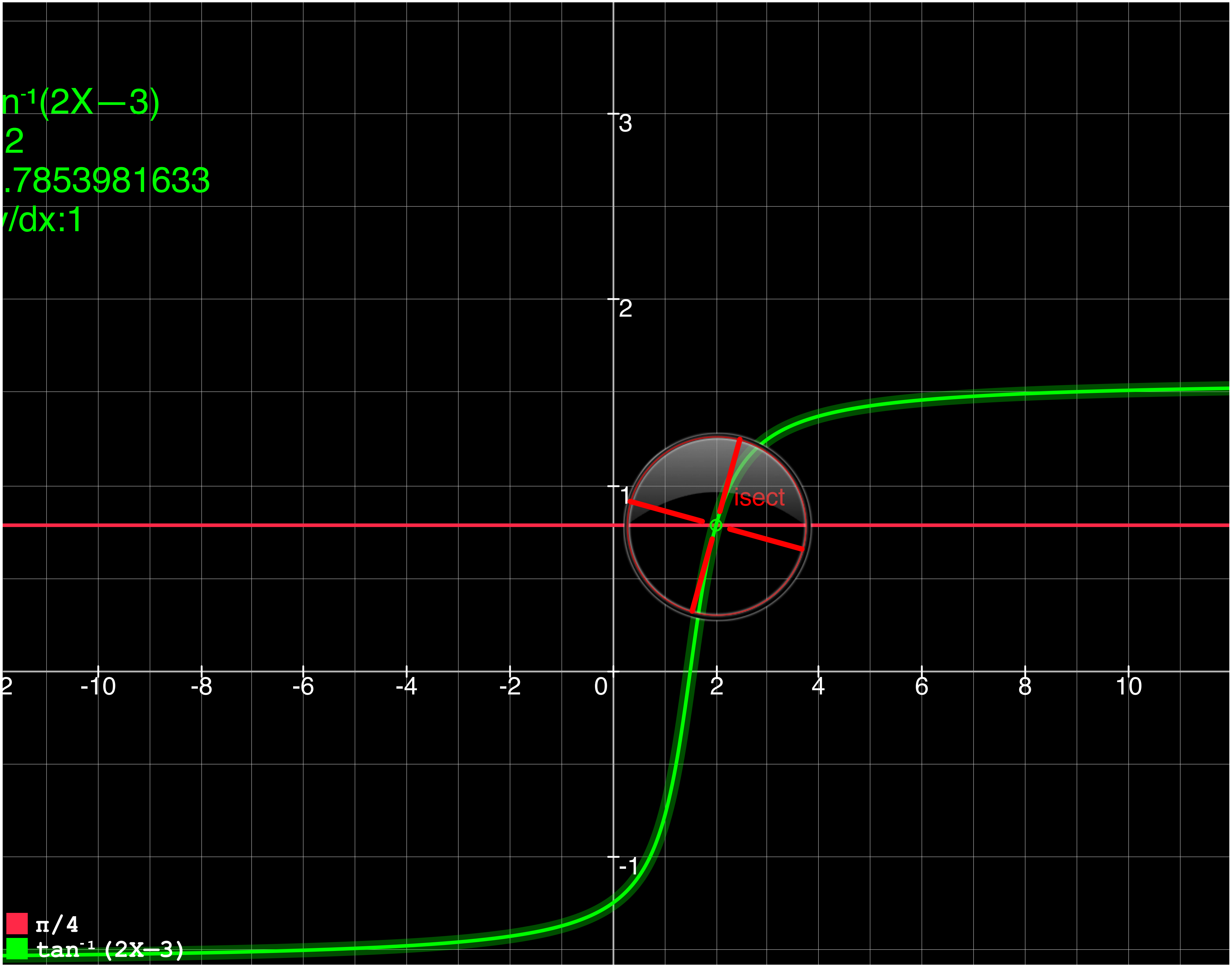
$$2x-3 = 1$$

$$2x-3+3 = 1+3$$

$$2x = 4$$

$$\frac{\cancel{2}x}{\cancel{2}} = \frac{4}{2}$$

$$x = 2$$



n⁻¹(2X—3)
2
.7853981633
/dx:1
3
2
1
isect
-1
2
-10
-8
-6
-4
-2
0
2
4
6
8
10
π/4
tan⁻¹(2X—3)

SOH-CAH-TOA

SOH - $\sin(y) = \frac{opp}{hyp}$

CAH - $\cos(y) = \frac{adj}{hyp}$

TOA - $\tan(y) = \frac{opp}{adj}$

$\csc(y) = \frac{hyp}{opp}$

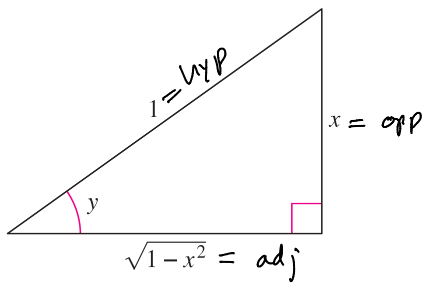
$\sec(y) = \frac{hyp}{adj}$

$\cot(y) = \frac{adj}{opp}$

Ex.3 Using Right Triangles

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.

b. Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.



$y = \arcsin x$

Figure 5.30

$y = \arcsin(x)$

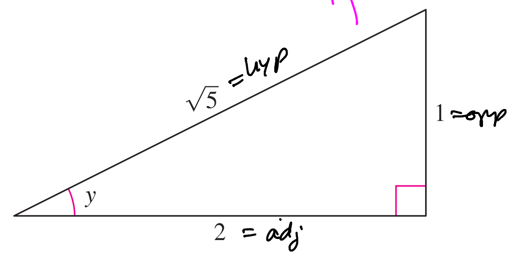
$\sin(y) = \sin[\arcsin(x)]$

$\sin(y) = x = \frac{x}{1} = \frac{opp}{hyp}$

$\cos(y) = \frac{adj}{hyp}$

$\cos(y) = \frac{\sqrt{1-x^2}}{1}$

$\cos(y) = \sqrt{1-x^2}$



$y = \operatorname{arcsec}\frac{\sqrt{5}}{2}$

Figure 5.31

$\sec(y) = \sec[\operatorname{arcsec}(\frac{\sqrt{5}}{2})]$

$\sec(y) = \frac{\sqrt{5}}{2} = \frac{hyp}{adj}$

$\tan(y) = \frac{opp}{adj}$

$\tan(y) = \frac{1}{2}$

Derivatives of Inverse Trigonometric Functions

In Section 5.1 you saw that the derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$. You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The following theorem lists the derivatives of the six inverse trigonometric functions. Proofs for arcsin u and arccos u are given in Appendix A, and the rest are left as an exercise. (See Exercise 104.) Note that the derivatives of arccos u, arccot u, and arccsc u are the *negatives* of the derivatives of arcsin u, arctan u, and arcsec u, respectively.

on the Final

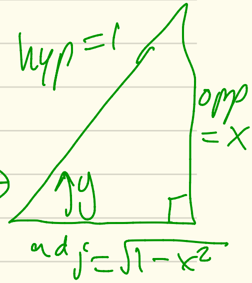
$$hyp^2 = opp^2 + adj^2$$
$$(1)^2 = x^2 + adj^2$$
$$1 - x^2 = adj^2$$
$$adj = \sqrt{1-x^2}$$

Derive: $\frac{d}{dx}[\arcsin(x)]$

Let $y = \arcsin(x)$

$$\sin(y) = \sin[\arcsin(x)]$$
$$\sin(y) = x$$

"SOH" $\sin(y) = \frac{x}{1} = \frac{opp}{hyp}$



$$\frac{d}{dx}[\sin(y)] = \frac{d}{dx}(x)$$

Implicit

$$\cos(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$

"CAH"

$$\cos(y) = \frac{adj}{hyp}$$
$$\cos(y) = \frac{\sqrt{1-x^2}}{1}$$
$$\cos(y) = \sqrt{1-x^2}$$

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$$

Let $y = \arctan(x)$

$\tan(y) = \tan[\arctan(x)]$

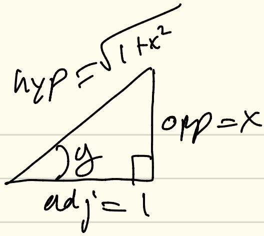
$\tan(y) = x$

$$\frac{d}{dx}\{\tan[y]\} = \frac{d}{dx}[x]$$

$$\sec^2(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$



$$\tan(y) = \frac{opp}{adj}$$

$$\tan(y) = \frac{x}{1}$$

$$hyp^2 = opp^2 + adj^2$$

$$hyp = \sqrt{1+x^2}$$

$$\sec(y) = \frac{hyp}{adj}$$

$$\sec(y) = \frac{\sqrt{1+x^2}}{1}$$

$$\sec(y) = \sqrt{1+x^2}$$

$$\sec^2(y) = 1+x^2$$

THEOREM 5.16 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

NOTE There is no common agreement on the definition of arcsec x (or arccsc x) for negative values of x. When we defined the range of the arcsecant, we chose to preserve the reciprocal identity

$$\text{arcsec } x = \arccos \frac{1}{x}.$$

For example, to evaluate arcsec(-2), you can write

$$\text{arcsec}(-2) = \arccos(-0.5) \approx 2.09.$$

One of the consequences of the definition of the inverse secant function given in this text is that its graph has a positive slope at every x-value in its domain. (See Figure 5.29.) This accounts for the absolute value sign in the formula for the derivative of arcsec x.

TECHNOLOGY If your graphing utility does not have the arcsecant function, you can obtain its graph using

$$f(x) = \text{arcsec } x = \arccos \frac{1}{x}.$$

Chain Rule

Ex.4 Differentiating Inverse Trigonometric Functions

a. $\frac{d}{dx}[\arcsin(2x)] = \frac{1}{\sqrt{1-(2x)^2}} \cdot \frac{d}{dx}(2x)$

$= \frac{1}{\sqrt{1-4x^2}} \cdot 2$

$= \frac{2}{\sqrt{1-4x^2}}$

b. $\frac{d}{dx}[\arctan(3x)] = \frac{1}{1+(3x)^2} \cdot \frac{d}{dx}[3x]$

chain Rule

$= \frac{1}{1+9x^2} \cdot 3$

$= \frac{3}{1+9x^2}$

$\sqrt{x} = x^{1/2}$

c. $\dfrac{d}{dx}\left[\arcsin \sqrt{x}\right] = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \dfrac{d}{dx}[x^{1/2}]$

$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{1-x}} \cdot \frac{1}{x^{1/2}}$$

$$= \frac{1}{2\sqrt{1-x}} \cdot \frac{1}{\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x-x^2}}$$

$$\boxed{\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}}}$$

chain

d. $\dfrac{d}{dx}\left[\operatorname{arcsec} e^{2x}\right] = \dfrac{1}{|e^{2x}|\sqrt{(e^{2x})^2-1}} \cdot \dfrac{d}{dx}[e^{2x}]$

$$= \frac{1}{e^{2x}\sqrt{e^{4x}-1}} \cdot e^{2x} \cdot \underbrace{\frac{d}{dx}(2x)}_{\text{chain Rule}}$$

$$= \frac{1}{\sqrt{e^{4x}-1}} \cdot 2$$

$$= \frac{2}{\sqrt{e^{4x}-1}}$$

Ex.5 A Derivative That Can Be Simplified

$$y = \arcsin x + x\sqrt{1-x^2}$$

Product Rule

Find $y' = \dfrac{d}{dx}[\sin^{-1}(x)] + \dfrac{d}{dx}[x \cdot (1-x^2)^{1/2}]$

$$y' = \frac{1}{\sqrt{1-x^2}} + x \cdot \frac{d}{dx}[(1-x^2)^{1/2}] + (1-x^2)^{1/2} \cdot \frac{d}{dx}(x)$$

$$y' = \frac{1}{\sqrt{1-x^2}} + x \cdot \frac{1}{2} \cdot (1-x^2)^{-1/2} \cdot \underbrace{(-2x)}_{\text{chain Rule}} + (1-x^2)^{1/2} \cdot 1$$

$$y' = (1-x^2)^{-1/2} - x^2 \cdot (1-x^2)^{-1/2} + (1-x^2)^{1/2}$$

$$y' = (1-x^2)^{-1/2}\left[1 - x^2 + (1-x^2)^{2/2}\right]$$

$$y' = (1-x^2)^{-1/2}\left[2 - 2x^2\right] \qquad (1-x^2)^{2/2} = 1-x^2$$

$$y' = \frac{2 \cdot [1-x^2]}{\sqrt{1-x^2}} = \underline{\underline{2\sqrt{1-x^2}}}$$

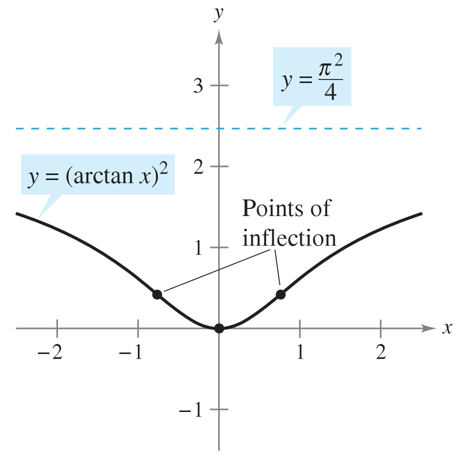
NOTE From Example 5, you can see one of the benefits of inverse trigonometric functions—they can be used to integrate common algebraic functions. For instance, from the result shown in the example, it follows that

$$\int \sqrt{1 - x^2}\, dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1 - x^2}\right).$$

■

Ex.6 Analyzing an Inverse Trigonometric Graph

Analyze the graph of $y = (\arctan x)^2$.



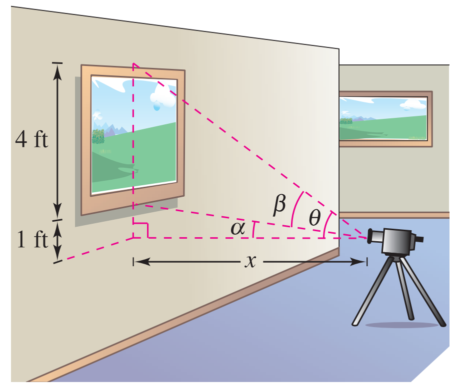
The graph of $y = (\arctan x)^2$ has a horizontal asymptote at $y = \pi^2/4$.
Figure 5.32

Ex.7 Maximizing an Angle

A photographer is taking a picture of a painting hung in an art gallery. The height of the painting is 4 feet. The camera lens is 1 foot below the lower edge of the painting, as shown in Figure 5.33. How far should the camera be from the painting to maximize the angle subtended by the camera lens?

Solution In Figure 5.33, let β be the angle to be maximized.

$$\begin{aligned}\beta &= \theta - \alpha \\ &= \operatorname{arccot} \frac{x}{5} - \operatorname{arccot} x\end{aligned}$$



Not drawn to scale

The camera should be 2.236 feet from the painting to maximize the angle β.
Figure 5.33

Review of Basic Differentiation Rules

As mathematics has developed during the past few hundred years, a small number of elementary functions have proven sufficient for modeling most* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	*Transcendental Functions*
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

With the differentiation rules introduced so far in the text, you can differentiate *any* elementary function. For convenience, these differentiation rules are summarized below.

Basic Differentiation Rules for Elementary Functions

1. $\dfrac{d}{dx}[cu] = cu'$	**2.** $\dfrac{d}{dx}[u \pm v] = u' \pm v'$	**3.** $\dfrac{d}{dx}[uv] = uv' + vu'$
4. $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$	**5.** $\dfrac{d}{dx}[c] = 0$	**6.** $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\dfrac{d}{dx}[x] = 1$	**8.** $\dfrac{d}{dx}[\lvert u \rvert] = \dfrac{u}{\lvert u \rvert}(u'), \quad u \neq 0$	**9.** $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$
10. $\dfrac{d}{dx}[e^u] = e^u u'$	**11.** $\dfrac{d}{dx}[\log_a u] = \dfrac{u'}{(\ln a)u}$	**12.** $\dfrac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\dfrac{d}{dx}[\sin u] = (\cos u)u'$	**14.** $\dfrac{d}{dx}[\cos u] = -(\sin u)u'$	**15.** $\dfrac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$	**17.** $\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u'$	**18.** $\dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1 - u^2}}$	**20.** $\dfrac{d}{dx}[\arccos u] = \dfrac{-u'}{\sqrt{1 - u^2}}$	**21.** $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$
22. $\dfrac{d}{dx}[\text{arccot } u] = \dfrac{-u'}{1 + u^2}$	**23.** $\dfrac{d}{dx}[\text{arcsec } u] = \dfrac{u'}{\lvert u \rvert\sqrt{u^2 - 1}}$	**24.** $\dfrac{d}{dx}[\text{arccsc } u] = \dfrac{-u'}{\lvert u \rvert\sqrt{u^2 - 1}}$

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6. $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\dfrac{d}{dx}[x] = 1$
8. $\dfrac{d}{dx}[|u|] = \dfrac{u}{|u|}(u'), \quad u \neq 0$
9. $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$
10. $\dfrac{d}{dx}[e^u] = e^u u'$
11. $\dfrac{d}{dx}[\log_a u] = \dfrac{u'}{(\ln a)u}$
12. $\dfrac{d}{dx}[a^u] = (\ln a)a^u u'$
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14. $\dfrac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\dfrac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1 - u^2}}$
20. $\dfrac{d}{dx}[\arccos u] = \dfrac{-u'}{\sqrt{1 - u^2}}$
21. $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$
22. $\dfrac{d}{dx}[\text{arccot } u] = \dfrac{-u'}{1 + u^2}$
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